Here are the examples that Quimby did in class 9 Nov 2006 to show the lower case letters i and l are not homeomorphic ($\not\approx$)

Let $i = [0,1] \cup \{2\}$ and $l = [0,2]$ Suppose $i \approx l$, then $\exists f : i \to l$, where f is bijective, f, f^{-1} is continuous. In particular $[0,1]$ is open in i, $\{2\}$ is open in i and $[0,1] \cup \{2\} = i$

Claim: $f([0,1])$ open in l

Proof: $f([0,1]) = (f^{-1})^{-1}([0,1])$ is open in l since f^{-1} is continuous.

Similarly $f(\{2\}) = (f^{-1})^{-1}$ is open in l

$f([0,1]) \cap f(\{2\}) = \varnothing$ and $f([0,1]) \cup f(\{2\}) = l$ since f is bijective.

$f([0,1]) \neq \varnothing$ and $f(\{2\}) \neq \varnothing$ therefore $(f([0,1]), f(\{2\}))$ is a separation of l.

But $l = [0,2]$ is a closed interval in $\mathbb{R}$, a contradiction, hence $i \not\approx l$

If one were to claim connected, explain why. For example $o \not\approx l$

$l = [0,2]$

$o = \{(\cos\theta, \sin\theta) \in \mathbb{R}^2 |\ 0 \leq \theta \leq 2\pi\}$ in the subspace topology

Remark: any point removed from o would leave something connected, but in l there are only two points that have this property

Let $p = f^{-1}(1)$

Claim: $l - \{1\}$ is not connected. Proof: $l = [0,1) \cup (1,2]$ is a separation of l

Consider $f^{-1}(l - \{1\}) = o - \{p\}$

You can prove (as you did last test) that $o - \{p\} \approx (0,1)$ (for Test 2, we can relax about explaining why this is so, and just use this conclusion)

(0,1) is connected.

$f|_{o-\{p\}}$ is a homeomorphism from $o - \{p\}$ to $l - \{1\}$

But $o - \{p\}$ is connected and $l - \{1\}$ is not connected, a contradiction.

Challenge problem: Show $Q \approx R$ Show $\nexists f : \mathbb{R}^2 \to \mathbb{R}^2$ homeomorphism such that $f(Q) = R$