Fall 2006 Math 530 Midterm #1

Do three (3) of the following topology questions. If you do all four, I'll count the best three.

#1) Let:

$$\mathsf{O} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

$$\mathsf{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1, x \geq 0\} \cup \{(0, y) \in \mathbb{R}^2 : -1 \leq y \leq 1\}$$

Sketch a proof of the fact that O is homeomorphic to D. Here are some claims you can make without proving them (as long as you're correct):

- That some set you've defined is closed in $\mathbb{R}^2$
- That some function you've defined is bijective (one-to-one and onto)
- That functions which you know from Calculus to be continuous are in fact continuous
- That some function you've defined is the inverse of some other function
- That Neptune is not a planet, but Pluto is

#2) We have seen in class (and so you do not need to prove) that:

$$\mathcal{Z} = \{\mathbb{R}^2 - Z(f) \mid f : \mathbb{R}^2 \to \mathbb{R} \text{ is a polynomial}\}$$

is a basis for a topology on $\mathbb{R}^2$ (called the Zariski topology), where we define $Z(f) := \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\}$. Prove in gory detail that the usual topology on $\mathbb{R}^2$ is finer than the Zariski topology. You may assume that every polynomial is a continuous function from $\mathbb{R}^2$ (in the usual topology) to $\mathbb{R}$ (in the usual topology).

#3) Let X be an ordered set.

(a) Suppose that X satisfies the *Intermediate Value Property*: "If $a, b \in X$ and $a < b$, then $\exists\, c \in X$ such that $a < c < b$." Prove in gory detail that if $a < b$, then $\overline{(a, b)} = [a, b]$. (I.e., show that the closure of an open interval is a closed interval.)

(b) Give an example of an ordered set X and two elements $a, b \in X$ such that $\overline{(a, b)} \neq [a, b]$. (Note that X must not satisfy the Intermediate Value Property.)

#4) Suppose X and Y are Hausdorff spaces. Let $\{x_n\}_{n \in \mathbb{Z}^+}$ and $\{y_n\}_{n \in \mathbb{Z}^+}$ be sequences in X and Y, respectively. Suppose that $x_n \to x \in X$ and $y_n \to y \in Y$ and $(x_n, y_n) \to (a, b) \in X \times Y$. Prove in gory detail that $x = a$ and $y = b$.

BONUS: What living individual has received the most Academy Award nominations?