

Fall 2006 Math 530 Midterm #1 Solutions

Do three (3) of the following topology questions. If you do all four, I'll count the best three.

#1) Let:

$$\mathcal{O} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

$$\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1, x \geq 0\} \cup \{(0, y) \in \mathbb{R}^2 : -1 \leq y \leq 1\}$$

Sketch a proof of the fact that \mathcal{O} is homeomorphic to \mathcal{D} . Here are some claims you can make without proving them (as long as you're correct):

- That some set you've defined is closed in \mathbb{R}^2
- That some function you've defined is bijective (one-to-one and onto)
- That functions which you know from Calculus to be continuous are in fact continuous
- That some function you've defined is the inverse of some other function
- That Neptune is not a planet, but Pluto is

Define $f_1 : \{(x, y) \in \mathcal{O} : x \leq 0\} \rightarrow \mathcal{D}$ by $f_1(x, y) = (0, y)$. Define $f_2 : \{(x, y) \in \mathcal{O} : x \geq 0\} \rightarrow \mathcal{D}$ by $f_2(x, y) = (x, y)$.

Define $f : \mathcal{O} \rightarrow \mathcal{D}$ by $f(x, y) = f_1(x, y)$ if $x \leq 0$ and $f(x, y) = f_2(x, y)$ if $x \geq 0$.

I claim that f is a homeomorphism. First, let's show that f is continuous.

$\{(x, y) \in \mathcal{O} : x \leq 0\}$ and $\{(x, y) \in \mathcal{O} : x \geq 0\}$ are both closed in \mathcal{O} .

$\{(x, y) \in \mathcal{O} : x \leq 0\} \cap \{(x, y) \in \mathcal{O} : x \geq 0\} = \{(0, 1), (0, -1)\}$.

$f_1(0, 1) = f_2(0, 1)$ and $f_1(0, -1) = f_2(0, -1)$.

f_1 and f_2 are continuous.

By the Pasting Lemma, f is continuous.

Also, f is one-to-one and onto.

All that remains is to show that f^{-1} is continuous.

Define $g_1 : \{(x, y) \in \mathcal{D} : x = 0\} \rightarrow \mathcal{O}$ by $g_1(0, y) = (-\sqrt{1 - y^2}, y)$. Define $g_2 : \{(x, y) \in \mathcal{D} : x \geq 0\} \rightarrow \mathcal{O}$ by $g_2(x, y) = (x, y)$.

Define $g : \mathcal{O} \rightarrow \mathcal{D}$ by $g(x, y) = g_1(x, y)$ if $x \leq 0$ and $g(x, y) = g_2(x, y)$ if $x \geq 0$.

An argument similar to the one above shows that g is continuous.

Since $g = f^{-1}$, we are done.

#2) We have seen in class (and so you do not need to prove) that:

$$\mathcal{Z} = \{\mathbb{R}^2 - Z(f) \mid f : \mathbb{R}^2 \rightarrow \mathbb{R} \text{ is a polynomial}\}$$

is a basis for a topology on \mathbb{R}^2 (called the Zariski topology), where we define $Z(f) := \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\}$. Prove in gory detail that the usual topology on \mathbb{R}^2 is finer than the Zariski topology. You may assume that every polynomial is a continuous function from \mathbb{R}^2 (in the usual topology) to \mathbb{R} (in the usual topology).

Let U be open in the Zariski topology. Show that U is open in the usual topology.

By the definition of a topology generated by a basis, $U = \cup_{\alpha \in J} U_\alpha$ for some set J and some collection $\{U_\alpha\}$ of elements of \mathcal{Z} .

So it suffices to show that for all $\alpha \in J$, we have that U_α is open in the usual topology.

Equivalently, we can show that $\mathbb{R}^2 - U_\alpha$ is closed in the usual topology.

Since $U_\alpha \in \mathcal{Z}$, we have that $\mathbb{R}^2 - U_\alpha = Z(f)$ for some polynomial f . Note that $Z(f) = f^{-1}(\{0\})$. So $\mathbb{R}^2 - U_\alpha = f^{-1}(\{0\})$. Since f is continuous, it now suffices to show that $\{0\}$ is closed in \mathbb{R} in the usual topology.

I claim that $\{0\}$ is closed in \mathbb{R} in the usual topology. To show this, we must show that $\mathbb{R} - \{0\}$ is open. To show this, it suffices to show that $(-\infty, 0)$ is open and $(0, \infty)$ is open, since $\mathbb{R} - \{0\} = (-\infty, 0) \cup (0, \infty)$. We know that $(-\infty, 0)$ is open, since it is the union of the basic open sets $(x, 0)$ for all negative real numbers x . Similarly, $(0, \infty)$ is open.

#3) Let X be an ordered set.

(a) Suppose that X satisfies the *Intermediate Value Property*: “If $a, b \in X$ and $a < b$, then $\exists c \in X$ such that $a < c < b$.” Prove in gory detail that if $a < b$, then $\overline{(a, b)} = [a, b]$. (I.e., show that the closure of an open interval is a closed interval.)

(b) Give an example of an ordered set X and two elements $a, b \in X$ such that $\overline{(a, b)} \neq [a, b]$. (Note that X must not satisfy the Intermediate Value Property.)

(a) Let $x \in \overline{(a, b)}$. Show $x \in [a, b]$.

$x \in (a, b) \cup (a, b)'$, where $(a, b)'$ denotes the set of all cluster points of the set (a, b) .

Case 1: $x \in (a, b)$. Then $x \in [a, b]$.

Case 2: $x \in (a, b)'$.

First we show that $x \geq a$. Temporarily assume that $x < a$. By the Intermediate Value Property, $\exists c \in X$ such that $x < c < a$.

Case 2a: x is a minimum of X .

Let $U = [x, a)$. Then U is a basic open set in X . Hence U is open in X . Also, $x \in U$. Also, $U \cap (a, b) = \emptyset$. But $V \cap (a, b) \neq \emptyset$ for all neighborhoods V of x . Contradiction.

Case 2b: x is not a minimum of X .

Then $\exists d \in X$ such that $d < x$.

Let $U = (d, a)$. Then U is a basic open set in X . Hence U is open in X . Also, $x \in U$. Also, $U \cap (a, b) = \emptyset$. But $V \cap (a, b) \neq \emptyset$ for all neighborhoods V of x . Contradiction.

The proof that $x \leq b$ is similar.

Therefore $x \in [a, b]$.

Now let $x \in [a, b]$. Show $x \in \overline{(a, b)}$.

Case 1: $x = a$

Show that $x \in (a, b)'$.

Let U be a basic neighborhood of x . Show that $U \cap (a, b) \neq \emptyset$.

Now either:

(i) $U = (c, d)$ for some $c, d \in X$ such that $c < d$, or

(ii) $U = (c, d]$ for some $c, d \in X$ such that $c < d$ and d is a maximum of X , or

(iii) $U = [c, d)$ for some $c, d \in X$ such that $c < d$ and c is a minimum of X .

I claim that $x < \min(b, d)$. We know $x < b$. In cases (i) and (iii) above, $x < d$ since $x \in U$.

In case (ii) above, we cannot have $x = d$, since $a < b$ implies that x is not a maximum of X ; therefore $x < d$ in this case, also. Therefore $x < \min(b, d)$.

So by the Intermediate Value Property, $\exists y \in X$ such that $x < y < \min(b, d)$. Then $y \in U \cap (a, b)$. So $U \cap (a, b) \neq \emptyset$.

Therefore $x \in (a, b)' \subset \overline{(a, b)}$.

Case 2: $a < x < b$

Then $x \in (a, b) \subset \overline{(a, b)}$.

Case 3: $x = b$

Similar to Case 1.

(b) In the integers \mathbb{Z} , the closure of $(-1, 1)$ is $\{0\}$.

#4) Suppose X and Y are Hausdorff spaces. Let $\{x_n\}_{n \in \mathbb{Z}^+}$ and $\{y_n\}_{n \in \mathbb{Z}^+}$ be sequences in X and Y , respectively. Suppose that $x_n \rightarrow x \in X$ and $y_n \rightarrow y \in Y$ and $(x_n, y_n) \rightarrow (a, b) \in X \times Y$. Prove in gory detail that $x = a$ and $y = b$.

I claim that $x_n \rightarrow a$ in X . Let U be a neighborhood of a in X . Show that $\exists N \in \mathbb{Z}^+$ such that if $k \geq N$, then $x_k \in U$.

We know that Y is open in Y . By the definition of the product topology, $U \times Y$ is open in $X \times Y$. Since $(x_n, y_n) \rightarrow (a, b) \in X \times Y$, we know that $\exists N \in \mathbb{Z}^+$ such that if $k \geq N$, then $(x_k, y_k) \in U \times Y$. Therefore, if $k \geq N$, then $x_k \in U$.

We have shown that $x_n \rightarrow a$ in X . We are given that $x_n \rightarrow x \in X$, and that X is Hausdorff. Therefore $x = a$.

The proof that $y = b$ is similar.

BONUS: What living individual has received the most Academy Award nominations?

John Williams.